Estimation of Probabilistic Minimum Inter-arrival Times Using Extreme Value Theory

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Abstract

Verifying timing constraints is the main purpose of analyses for real-time systems. This verification can be done using probabilistic methods, which rely on statistical estimations of certain task parameters. In this paper, we address the problem of determining probability distributions for minimum inter-arrival times of tasks, using statistical methods applied on data obtained through measurements.

1 Introduction

Nowadays Critical Real-Time Embedded Systems (CRTESs) are prevalent in many sectors and it is estimated that 98% of current processors are embedded. The performances of CRTESs are analyzed not only from the point of view of their correctness, but also from the perspective of time. Moreover, the increasing need for new functionalities imposes the use of modern and complex architectures. These architectures have a direct impact on the time variability of programs and applications exploiting functional and non-functional behavior of the CRTESs.

The timing analysis of such systems has been extensively studied by considering deterministic approaches based on worst-case scenarios that induce a certain pessimism. Unfortunately not all real-time systems can afford this pessimism and the consequent over-provisioning, and for these cases other approaches should be considered. Alternative approaches could be statistical and probabilistic, agent learning, game theory, etc. In this paper we are interested in the statistical and probabilistic approaches. These approaches offer an accurate and enriched representation of the parameters of the CRTESs. For example, a task parameter can be described as a random variable or the function expressing the resource given to a task flow can be modeled in terms of probabilistic bounds.

2 Motivation, Model and Notations

Probabilistic real-time systems and probabilistic real-time analysis is becoming a common practice in the real-time community, [1]. Papers related to this topic use different terms like *stochastic analyses* [2–4], *probabilistic analyses* [5] or *statistical analysis* [6, 7] to indicate usually that the considered CRTES has at least one parameter defined by a random variable. In this paper we make use of the word *statistic* to indicate that the work is based on the theory of statistics and the word of *probabilistic* to indicate that the work is based on the theory of probability.

The system that we are analysing consists of n constrained deadline, sporadic tasks $\{\tau_1, \tau_2, \ldots, \tau_n\}$. Each task τ_i is characterized by three parameters $(C_i, \mathcal{T}_i, D_i)$ where C_i is the worst case execution time, D_i the relative deadline, and $\mathcal{T}_i$ the pMIT described by a random variable[1]. We assume a constrained-deadline task model such that $D_i \leq min(\mathcal{T}_i)$, where $min(\mathcal{T}_i)$ is the minimum value of $\mathcal{T}_i$.

We are interested in determining distributions of probabilistic minimum inter-arrival times (pMIT) and their computation. By pMIT we mean a random variable that provides the probability that the minimum inter-arrival time of a task is larger than a given value.

The classic, worst case analysis takes into consideration values that upper bound or lower bound the behavior of tasks, such as worst case execution time to upper bound the execution times, and minimum inter-arrival time to lower bound the inter-arrival times.

In reality, jobs can arrive with different inter-arrival times and the minimum one only occurs with a relatively low frequency. Also, the MIT value taken into consideration by the worst case analysis can be substantially smaller than the inter-arrival times that occur most frequent during the runtime of the system and that determine its average behaviour. This factors can lead to considerable pessimism in the analysis, leading to severe over-provisioning of the

[1] Calligraphic typeface is used to denote random variables.

system.

This pessimism can be decreased by taking into consideration all the possible values of inter-arrival times. Considering that a job $\tau_{i,j}$ of a task τ_i can arrive after *t* units of time from the arrival of the previous job of task τ_i, i.e., job $\tau_{i,j}$, where *t* can take values between the minimum inter-arrival time value and the maximum inter-arrival time value, then one can observe the arrivals of jobs and, through statistical methods, generate a distribution of the minimum inter-arrival times, from which *t* can take values with certain probabilities. To this scope we make use of the statistical software R, applying methods specific to the extreme value theory in order to verify if the observed distribution fits a target distribution, the Weibull distribution in our case.

3 Extreme Value Theory

The Theory of the Extreme Values (EVT) estimates the probability of occurrence of the extreme events. This theory studies the behavior of the upper and lower tails for sequences of random variables when their distribution function is unknown. EVT is based on asymptotic arguments for sequences of observations; it provides information about the distribution of the maximum value as the sequences size increases (see [8]).

Let $\mathcal{X}_1, .., \mathcal{X}_n$ be a sample of size n drawn from the variables $\mathcal{X}$. Its distribution function F is then given by:

$$F(x) \;=\; P(\mathcal{X}_i \leq x) \quad \text{for} \;\; i = 1, .., n \tag{1}$$

Without knowing the general statistical behavior of the sequence we would like to study its extreme behavior. We consider then the maximum of the sequence $\mathcal{M}_n$ denoted by: $\mathcal{M}_n = \max(\mathcal{X}_1, \mathcal{X}_2, .., \mathcal{X}_n)$ (we can treat the minimum in the same way using the correspondence between $\min(\mathcal{X}_1, \mathcal{X}_2, .., \mathcal{X}_n) = -\max(-\mathcal{X}_1, -\mathcal{X}_2, .., -\mathcal{X}_n)$, so all the results for the maximum can be transposed for minimum). The observations in the sample being i.i.d. then the distribution function of the maximum is:

$$F_{\mathcal{M}_n}(x) = P(\mathcal{M}_n \leq x) = P^n(\mathcal{X} \leq x) = F^n(x) \tag{2}$$

It is not possible to find this distribution without knowing the distribution function of the random variable X. Nevertheless, under certain assumptions we may find the asymptotic behavior of $\mathcal{M}_n$, for large value of n (the samples size). EVT theory provides information about the distribution of the maximum value of such an i.i.d. sample as n increases (see [9]).

Definition 1. *(Distributions of the same type). The distributions F and F^* are of the* ***same type*** *if there are constants $a > 0$ and b such that $F^*(ax+b) = F(x)$ for all x. Two random variables are of the same type if theirs distributions are of the same type. In other words, the variables of the same type have the same law in a factor of location and scale near.*

Similar to the central limit theorem (CLT), we can find normalization constants $a_n > 0$ and $b_n \in \mathbb{R}$ and a non-degenerate distribution H such as:

$$P\left\{\frac{\mathcal{M}_n - b_n}{a_n} \leq x\right\} = (F(a_n x + b_n))^n \rightarrow H(x), \text{for} \;\; n \rightarrow \infty \tag{3}$$

The foundations of the theory of extreme values are set by Fisher and Tippett which propose a first solution to the problem associated to Equation 3. The following theorem presenting this solution is often called *the first EVT theorem*.

Theorem 3.1. ***(Fisher-Tippett or Extremal Types Theorem)*** *Let $\mathcal{X}_1, \mathcal{X}_2, .., \mathcal{X}_n$ be independent random variables with the same probability distribution, and $\mathcal{M}_n = max(\mathcal{X}_1, \mathcal{X}_2, .., \mathcal{X}_n)$. If there exist sequences of constants $a_n > 0$ and b_n, such that, as $n \rightarrow \infty$, $Pr\left\{\frac{\mathcal{M}_n - b_n}{a_n} \leq x\right\} \rightarrow G(x)$ for some non-degenerate distribution G, then G has the same type as one of the following distributions:*

Type I (Gumbel)

$$G(x) = \exp\left\{-\exp\left(-\frac{x-b}{a}\right)\right\}, \;\; -\infty < x < \infty;$$

Type II (Fréchet)

$$G(x) = \begin{cases} 0, & x \leq b, \\ \exp\left(-\left(\frac{x-b}{a}\right)^{-\alpha}\right), & x > b; \end{cases}$$

Type III (Weibull)

$$G(x) = \begin{cases} \exp\left\{-\left(-\left(\frac{x-b}{a}\right)\right)^{\alpha}\right\}, & x < b \\ 1, & x \geq b. \end{cases}$$

for parameters $a > 0$, b and in case of families II and III, $\alpha > 0$.

Thus theorem 3.1 states that *if* the distribution of the rescaled maxima $\frac{\mathcal{M}_n - b_n}{a_n}$ converges, then the limit $G(x)$ is one of the three types, whatever the distribution of the variable parent (e.g. [10]).

Although the behavior of the three laws is completely different, they can be combined into a single parametrization containing one parameter ξ that controls the "heaviness" of the tail, called the shape parameter. This law is called the Generalized Extreme Value distribution (GEV) and its most general forme is obtained by introducing a location, μ and scale, σ parameters:

$$G(x) = \exp\left\{ -\left[1+\xi\left(\frac{x-\mu}{\sigma}\right)\right]_{+}^{-\frac{1}{\xi}} \right\} \tag{4}$$

The location parameter, μ determines where the distribution is concentrated, the scale parameter, σ determines its width. The shape parameter ξ determines the rate of tail decay (the larger ξ, the heavier the tail), with:

- $\xi > 0$ indicating the heavy-tailed (Fréchet) case
- $\xi = 0$ indicating the light-tailed (Gumbel, limit as $\xi \longrightarrow 0$) case
- $\xi < 0$ indicating the truncated distribution (Weibull) case

If we take into account the GEV, then the extremal theorem may be reformulated as follows: the asymptotic behavior of the maximum of a sufficiently large sample is a GEV distribution. In the same way as for the CLT, a max-stability property makes possible the convergence of the maxima and it allows to find the distribution it converges to.

Definition 2. *(Max-stability) A distribution G is said to be max-stable if a linear combination of two independent variables from the G distribution, has also a G distribution, up to affine transformations, i.e. up to location and scale parameters.*

So in this case we would like to find what type of distributions are stable for the maxima $\mathcal{M}_n$ up to affinity, i.e. the distributions satisfying: $\mathcal{M}_n = max(\mathcal{X}_1, .., \mathcal{X}_n) \stackrel{d}{=} a_n\mathcal{X}_i + b_n$ ($\mathcal{X} \stackrel{d}{=} \mathcal{Y}$ means that the two random variables $\mathcal{X}$ and $\mathcal{Y}$ are equal in distribution) for the sample-size-dependent scale and location parameters $a_n > 0$ and b_n and $\mathcal{X}_i$ from the parent distribution.

The GEV theory is used for the block maxima approach i.e. only the maximum value of the data within a certain time interval are considered to be extreme values. In practice this approach may have some restrictions and an alternative approach avoiding this inconvenient is the Peaks Over Thresholds (POT) that consists in modeling exceedances above a pre-chosen threshold.

4 Example of possible distribution for the probabilistic minimal inter-arrival time

We present within this section an example of distribution for the pMIT. This distribution is obtained from observed data and, according to EVT, this distribution follows one of the three distributions. Since the distribution of pMIT describes minimum values, then the expected distribution is Weibull.

The empirical data are obtained by observation of the instants of the arrival of a task τ_i. These data are then fitted to the closest Weibull distribution and the obtained Weibull distribution describes the pMIT of τ_i.

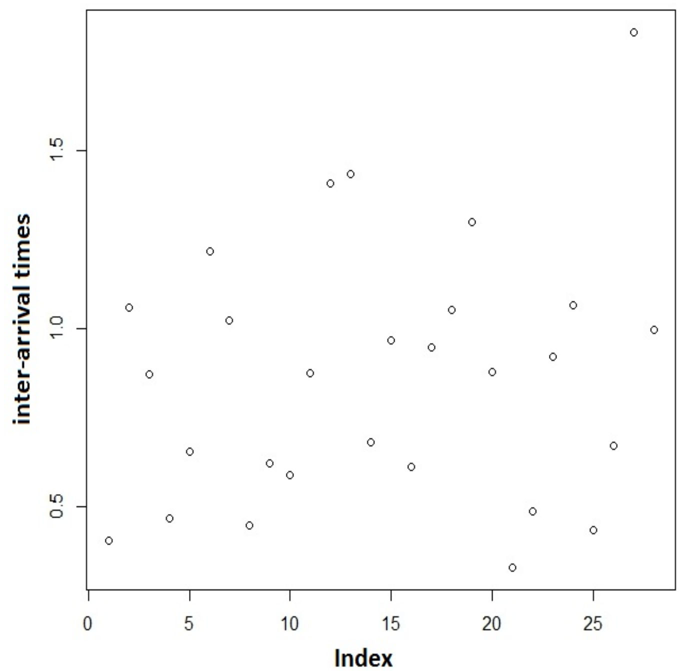

Figure 1. The distribution of the observed inter-arrival times of task τ_i

For instance we consider the observed data described by the set $A = \{0.403971118683603, 1.05972752792893, 0.873661257610701, 0.466451579211314, 0.65681133533767, 1.21913083058972, 1.02279223624766, 0.448211923626033, 0.622974878754791, 0.58927823670255, 0.875802550071033, 1.40745617870966, 1.43475364370752, 0.682358221286792, 0.966765792647443, 0.61305654030804, 0.946864164882801, 1.05210241861247, 1.30080188489985, 0.880415865075983, 0.328982288841207, 0.486749488133052, 0.9229513562478591.06578265441239, 0.435684506376465, 0.672950959519735, 1.83491415176197, 0.998400602399684\}$.

The set A may be visualized in Figure 1. We compare these data visually to the Weibull distribution of shape 2 and scale 1. Figure 2 indicates that these data may follow a Weibull distribution. This intuition is confirmed formally by using the Kolmogorov-Smirnov test that compares two distributions by calculating the corresponding p-value. If the p-value is larger than 0.05, then the two distributions are sufficiently close to conclude that our data belong to the Weibull domain.

We run the Kolmogorov-Smirnov test using the language R and the command:

ks.test(A,"pweibull", shape=2,scale=1)

We obtain the p-value= 0.5711 indicating that our data belong to the Weibull domain.

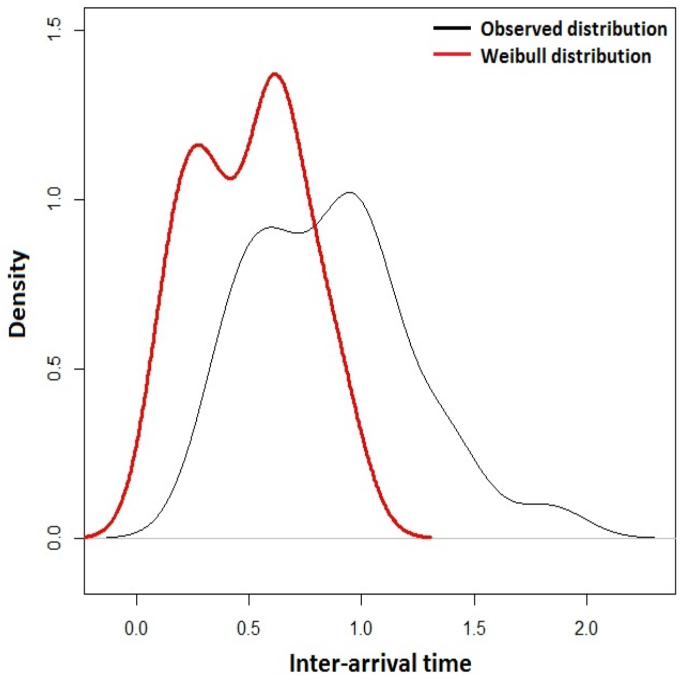

Figure 2. Visual comparison of the empirical distribution and the Weibull distribution

5 Conclusions and future work

We present a first work on obtaining the probabilistic minimal inter-arrival time of a task. This work is based on the utilization of the Weibull distribution, one of the three distributions of the Extreme Value Theory. Usually this distribution is used to estimate the minimum values while using the Extreme Value Theory. Here we present a first step for such use, but future work is needed to propose a methodology for finding the parameters of the Weibull distribution. Moreover, in some cases the utilization of the GEV distribution may be necessary as it is the case of the estimation of the probabilistic response time [11].

References

[1] A. Burns, G. Bernat, and I. Broster, "A probabilistic framework for schedulability analysis," in *Third International Embedded Software Conference (EMSOFT 2003)*, 2003, pp. 1–15.

[2] M. Gardner and J. Lui, "Analyzing stochastic fixed-priority real-time systems," in *5thInternational Conference on Tools and Algorithms for the Construction and Analysis of Systems*, 1999.

[3] G. A. Kaczynski, L. L. Bello, and T. Nolte, "Towards stochastic response-time of hierarchically scheduled real-time tasks," in *Proceedings of 11th IEEE International Conference on Emerging Technologies and Factory Automation (ETFA 2006)*, 2006, pp. 453–456.

[4] J. L. Díaz, D. F. García, K. Kim, C.-G. Lee, L. Lo Bello, J. M. López, S. L. Min, and O. Mirabella, "Stochastic analysis of periodic real-time systems," in *Proceedings of the 23rd IEEE Real-Time Systems Symposium (RTSS 2002)*, 2002, p. 289.

[5] T. Tia, Z. Z. Deng, M. Shankar, M. Storch, J. Sun, L. Wu, and J. Liu, "Probabilistic performance guarantee for real-time tasks with varying computation times." in *IEEE Real-Time and Embedded Technology and Applications Symposium (ETFA 1995)*, 1995.

[6] A. Atlas and A. Bestavros, "Statistical rate monotonic scheduling," in *19th IEEE Real-Time Systems Symposium (RTSS 1998)*, 1998.

[7] L. Cucu-Grosjean, L. Santinelli, M. Houston, C. Lo, T. Vardanega, L. Kosmidis, J. Abella, E. Mezzeti, Q. E., and F. Cazorla, "Measurement-based probabilistic timing analysis for multi-path programs," in *the 24th Euromicro Conference on Real-Time Systems (ECRTS12)*, 2012.

[8] S. COLES, *An Introduction to Statistical Modeling of Extreme Values*. Springer, 2001.

[9] M. GHIL, P. YIOU, S. HALLEGATTE, B. MALAMUD, and P.NAVEAU, "Extreme events: dynamics, statistics and prediction," Copernicus Publications, Tech. Rep., 2011.

[10] B. RAGGAD, "Fondements de la thorie des valeurs extrmes, ses principales applications et son apport la gestion des risques du march ptrolier," *Mathematics and Social Sciences*, vol. 186, pp. 29–63, 2009.

[11] I. B. Y. Lu, T. Nolte and L. Cucu-Grosjean, "A statistical response-time analysis of real-time embedded systems," in *IEEE Real-Time Systems Symposium (RTSS2012)*, 2012.